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On the qualitative behavior of periodic solutions of differential systems ☆

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ABSTRACT

This article deals with the reflective function of differential systems. The obtained results are applied to studying the existence and stability of the periodic solutions of some linear and nonlinear periodic differential systems.

1. Introduction

As we know, studying the property of the solutions of the differential system

$$x' = X(t, x). \tag{1}$$

is very important not only for the theory of ordinary differential equation but also for practical reasons. If $X(t + 2\omega, x) = X(t, x)$ (ω is a positive constant), to study the behavior of solutions of (1), we could use, as introduced in [1,2], the Poincaré mapping. But it is very difficult to find the Poincaré mapping for many systems which cannot be integrated in quadratures. In the 1980s the Russian mathematician Mironenko [2] first established the theory of reflective functions (**RF**). Then a quite new method to study (1) was found.

In the present section, we introduce the concept of the reflective function, which will be used throughout the rest of this article.

Now consider the system (1) with a continuously differentiable right-hand side and with a general solution $\psi(t; t_0, x_0)$. For each such system, the **reflective function (RF)** of (1) is defined as $F(t, x) := \psi(-t; t, x)$. Then for any solution $x(t)$ of (1), we have $F(t, x(t)) = x(-t)$. If system (1) is 2ω-periodic with respect to t, and F is its **RF**, then $F(-\omega, x) = \psi(\omega; -\omega, x)$ is the Poincaré [1,2] mapping of (1) over the period $[-\omega, \omega]$. So, for any solution $x(t)$ of (1) defined on $[-\omega, \omega]$, it will be 2ω-periodic if and only if $x(-\omega)$ is a fixed point of the Poincaré mapping $T(x) = F(-\omega, x)$.

A function $F(t, x)$ is a **RF** of system (1) if and only if it is a solution of the partial differential equation

$$F'_t + F'_x X(t, x) + X(-t, F) = 0$$

with the initial condition $F(0, x) = x$. This implies that for non-integrable infinite terms periodic systems we can also find out their Poincaré mapping. If, for example, $X(t, x) + X(-t, x) \equiv 0$, then $T(x) \equiv F(-\omega, x) = x$.

Each continuously differentiable function F that satisfies the condition

$$F(-t, F(t, x)) \equiv F(0, x) \equiv x$$

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is a **RF** of the whole class of systems of the form [2]

$$x' = -\frac{1}{2}\frac{\partial F}{\partial x}(-t, F(t,x))\left(\frac{\partial F(t,x)}{\partial t} - 2S(t,x)\right) - S(-t, F(t,x)), \tag{2}$$

where S is an arbitrary vector function such that solutions of system (2) are uniquely determined by their initial conditions.

Therefore, all systems of the form (1) are split into equivalence classes of the form (2) so that each class is specified by a certain reflecting function referred to as the **RF** of the class.

For all systems of one class, the shift operator [1,2] on the interval $[-\omega, \omega]$ is the same. Therefore, all equivalent 2ω-periodic systems have a common mapping over the period, and the behaviors of the periodic solutions of these systems are the same.

By the literature [2], we know that if $F(t,x)$ is the **RF** of system (1), then it is also the **RF** of system

$$x' = X(t,x) + F_x^{-1}(t,x)R(t,x) - R(-t, F(t,x)),$$

where R is an arbitrary continuously differentiable function on $R \times R^n$.

By [3], if $\Delta(t,x)$ is a solution of the partial differential equation

$$\Delta_t' + \Delta_x' X(t,x) - X_x'(t,x)\Delta = 0, \tag{3}$$

then the **RF** of system (1) and its perturbed system

$$x' = X(t,x) + \alpha_o(t)\Delta(t,x) \tag{4}$$

coincide, where $\alpha_o(t)$ is an arbitrary odd scalar function.

Let system (1) be linear, i.e.,

$$x' = A(t)x, \quad t \in R, x \in R^n, \tag{5}$$

and $\Phi(t)$ is its the fundamental matrix of solutions such that $\Phi(0) = E$, where E is the $n \times n$ unit matrix. Then the general solution of (5) is $x = \varphi(t; t_0, x_0) = \Phi(t)\Phi^{-1}(t)x_0$. Therefore, the **RF** of (5) is linear and $F(t,x) = F(t)x$, where $F(t) = \Phi(-t)\Phi^{-1}(t)$. This matrix $F(t)$ is referred to as a **reflective matrix** (**RM**) of system (5).

The **RM** of system (5) satisfies the relations $F(-t)F(t) \equiv F(0) = E$. Differentiable matrix $F(t)$ is a **RM** of system (5) if and only if it is a solution of the system

$$F'(t) + F(t)A(t) + A(-t)F(t) = 0$$

with the initial condition $F(0) = E$.

If matrix $A(t)$ is 2ω-periodic and $F(t)$ is **RM** of system (5), then for this system the matrix $F(-\omega)$ is similar to a monodromy matrix of (5) on the interval $[-\omega, \omega]$. Thus solutions μ_i $(i = 1, 2, \ldots, n)$ of the equation $\det(F(-\omega) - \mu E) = 0$ are the Floquet multipliers of system (5) [4].

By the preceding state and [5], if $F(t)$ is the **RM** of linear system (5), then it is also **RM** of the following systems

$$x' = A(t)x + F(-t)R(t,x) - R(-t, F(t)x), \tag{6}$$

$$x' = A(t)x + \alpha_o(t)\Delta(t,x), \tag{7}$$

$$x' = A(t)x + \sum_{i=0}^{m} \alpha_{oi}(t)B^i(t)x, \tag{8}$$

where $R(t,x)$ is the same as before, $\Delta(t,x)$ is a solution of (3) with $X(t,x) = A(t)x$, $B(t)$ is a matrix and satisfies the relation

$$B'(t) = A(t)B(t) - A(t)B(t) + \sum_{j=0}^{n} \beta_{oj}(t)B^j(t),$$

$\alpha_o(t), \alpha_{oi}(t), \beta_{oj}$ $(i = 0, 1, 2 \ldots m, j = 0, 1, 2, \ldots, n.)$ are continuously differentiable odd scalar functions.

Furthermore, if the systems (5)–(8) are 2ω-periodic with respect to t, then all these periodic systems have a common Poincaré mapping over the period $[-\omega, \omega]$, and the behavior of the periodic solutions of these systems are the same.

Thus, to study the qualitative behavior of solutions of periodic systems (5)–(8), it is necessary to find out the **RM** of linear system (5).

There exist a wide number of articles devoted to look for the special **RF**. Alisevich [6,7] obtained the criterion for the linear system (5) that the **RM** is in the diagonal or triangular form. Veresovich [8] and Zhou [9,10] established the necessary and sufficient conditions for the **RM** of system (5) satisfying identity $F(t)F^T(t) = \delta_o(t)E$. Musafirov [4,11] found out the **RM** of system (5) in the form of $F(t) = e^{\alpha_1(t)M}e^{\beta_1(t)N}e^{-\alpha_1(-t)M}$. They used the obtained results to discuss the qualitative analysis of solutions of system (5) and its equivalent systems and obtained a lot of good results. To apply **RF** more widely to discuss the properties of solutions of (1) is very important. There will be much work for us to do.

In this paper we will consider the linear system (5) and give the sufficient conditions for the **RM** of system (5) to be in the general form and discuss the existence and stability of the periodic solutions of systems (5)–(8). The present research extend the known conclusions.

In the following, let us denote $a = a(t)$; $\bar{a} = a(-t)$; $a_e = \frac{a+\bar{a}}{2}$; $a_o = \frac{a-\bar{a}}{2}$; $a^* = \frac{a-d}{2}$; $\operatorname{tr} A_e = a_e + d_e$. The notation $a \neq 0$ means that, in some deleted neighborhood of $t = 0$ and $|t|$ being small enough, a is different from zero.

2. Main results

Now, we consider the linear differential system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A(t) \begin{pmatrix} x \\ y \end{pmatrix}, \tag{9}$$

where $A(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a continuously differentiable matrix function in $\mathbf{R}$. Suppose that $F(t, x, y) = (f_1(t)x + f_2(t)y, f_3(t) + f_4(t)y)^{\mathrm{T}}$ is the **RF** of (4), in which, $b \neq 0, f_2 \neq 0$. In the special case: $b = 0$ or $f_2 = 0$, the **RF** of system (9) has been discussed in [6,7].

Lemma 1. *Matrix $F(t) = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$ is the **RM** of (9) if and only if $F(t) = \mathrm{e}^{\alpha(t)} G(t)$, where*

$$\alpha = -\int_0^t \operatorname{tr} A_e(t)\mathrm{d}t,$$

$$G(t) = \begin{pmatrix} g_1 & g_2 \\ g_3 & \bar{g}_1 \end{pmatrix}, \tag{10}$$

in which

$$g_1\bar{g}_1 - g_2 g_3 = 1, \qquad g_2 + \bar{g}_2 = 0, \qquad g_3 + \bar{g}_3 = 0.$$

Furthermore, $G(t)$ is the **RM** of system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a^* & b \\ c & -a^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{11}$$

Proof. By the preceding introduction, we know that $F(t)$ is the **RM** of (9) if and only if

$$\begin{cases} F' + FA + \bar{A}F = 0, \\ F(0) = E, \end{cases} \tag{12}$$

where $F(t)$ satisfies relations

$$\det F = \exp\left(-2\int_0^t \operatorname{tr} A_e \mathrm{d}t\right) = \mathrm{e}^{2\alpha}, \quad \alpha = -\int_0^t \operatorname{tr} A_e \mathrm{d}t, \tag{13}$$

$$F\bar{F} = E. \tag{14}$$

From (12) it follows that

$$\begin{cases} f_1' + f_1(a + \bar{a}) + f_2 c + f_3 \bar{b} = 0, \\ f_2' + f_2(\bar{a} + d) + f_1 b + f_4 \bar{b} = 0, \\ f_3' + f_3(a + \bar{d}) + f_4 c + f_1 \bar{c} = 0, \\ f_4' + f_4(d + \bar{d}) + f_3 b + f_2 \bar{c} = 0, \\ f_1(0) = f_4(0) = 1, \qquad f_2(0) = f_3(0) = 0. \end{cases} \tag{15}$$

Relation (13) implies

$$f_1 f_4 - f_2 f_3 = \mathrm{e}^{2\alpha}. \tag{16}$$

From (14), we have

$$\begin{cases} f_1\bar{f}_1 + f_2\bar{f}_3 = 1, \\ f_1\bar{f}_2 + f_2\bar{f}_4 = 0, \\ f_3\bar{f}_1 + f_4\bar{f}_3 = 0, \\ f_3\bar{f}_2 + f_4\bar{f}_4 = 1. \end{cases} \tag{17}$$

For the second equation in system (15), multiplying by f_1, subtracting from the first equation in system (15), multiplying f_2 and using (16), we obtain

$$f_2 f_1' - f_1 f_2' + (a - d) f_1 f_2 + c f_2^2 - b f_1^2 = \bar{b}\mathrm{e}^{2\alpha}. \tag{18}$$

From the first equation in system (15) solve for f_3, from the second equation in system (15) solve for f_4, and substitute them into the first and second equations in system (17). Then we have

$$\begin{cases} \bar{f}_2(f_1' + f_1(a+\bar{a}) + f_2 c) = \bar{b}f_1\bar{f}_1 - \bar{b}, \\ \bar{f}_2(f_2' + f_2(d+\bar{a}) + f_1 b) = \bar{b}\bar{f}_1 f_2. \end{cases}$$

These identities imply

$$\bar{f}_2(f_2f_1' - f_1f_2' + (a-d)f_1f_2 + cf_2^2 - bf_1^2) = -\bar{b}f_2. \tag{19}$$

Substituting (18) into (19) yields

$$f_2 e^{-\alpha} + \bar{f}_2 e^{\alpha} = 0.$$

Let denote $g_2 = f_2 e^{-\alpha}$. Then $g_2 + \bar{g}_2 = 0$,

$$f_2 = g_2 e^{\alpha}. \tag{20}$$

Substituting (20) into the second identity in (17), we obtain

$$g_2(\bar{f}_4 e^{\alpha} - f_1 e^{-\alpha}) = 0.$$

As $f_2 \neq 0$, $g_2 \neq 0$. Let $g_1 = f_1 e^{-\alpha}$. Then $g_1(0) = 1$,

$$f_1 = g_1 e^{\alpha}, \qquad f_4 = \bar{g}_1 e^{\alpha}. \tag{21}$$

Substituting (21) into the third identity in (17), we obtain

$$f_3 e^{-\alpha} + \bar{f}_3 e^{\alpha} = 0.$$

Denoting $g_3 = f_3 e^{-\alpha}$, then $g_3 + \bar{g}_3 = 0$, $f_3 = g_3 e^{\alpha}$. Therefore,

$$F = e^{\alpha}\begin{pmatrix} g_1 & g_2 \\ g_3 & \bar{g}_1 \end{pmatrix} = Ge^{\alpha}, \tag{22}$$

$\det G = g_1\bar{g}_1 - g_2g_3 = 1$. Substituting (22) into (15), we obtain $G(0) = E$ and

$$\begin{cases} g_1' = -2a_e^* g_1 - g_2 c - g_3\bar{b} = P_1, \\ g_2' = 2a_o^* g_2 - g_1 b - \bar{g}_1\bar{b} = P_2, \\ g_3' = -2a_o^* g_3 - c\bar{g}_1 - g_1\bar{c} = P_3, \\ \bar{g}_1' = 2a_e^*\bar{g}_1 - bg_3 - g_2\bar{c} = -\bar{P}_1, \end{cases} \tag{23}$$

where $a^* = \frac{a-d}{2}$, $a_e^* = \frac{a^*+\bar{a}^*}{2}$, $a_0^* = \frac{a^*-\bar{a}^*}{2}$. Thus, G is the **RM** of system (11). The proof is finished. □

By Lemma 1, to look for the **RM** F of system (9), it is necessary to seek the **RM** G of system (11). This will be done in what follows.

Remark 1. It is clear, relation (23) implies that matrix function (10) is the **RM** of system (11) if and only if

$$\begin{cases} g_{1o}' = -2a_e^* g_{1e} - g_2 c_o + g_3 b_o = Q_0, \\ g_{1e}' = -2a_e^* g_{1o} - g_2 c_e - g_3 b_e = Q_1, \\ g_2' = 2a_o^* g_2 - 2g_{1e}b_e - 2g_{1o}b_o = Q_2, \\ g_3' = -2a_o^* g_3 - 2g_{1e}c_e + 2g_{1o}c_o = Q_3, \\ g_{1e}(o) = 1, \end{cases} \tag{24}$$

where

$$g_{1e} = \frac{g_1 + \bar{g}_1}{2}, g_{1o} = \frac{g_1 - \bar{g}_1}{2}.$$

Remark 2. Using (24), we can obtain that if

$$a_e^* = c_o = b_o = 0,$$

then $G = \begin{pmatrix} g_{1e} & g_2 \\ g_3 & g_{1e} \end{pmatrix}$ is the **RM** of system (11) if and only if

$$\begin{cases} g_{1e}' = -g_2 c_e - g_3 b_e, \\ g_2' = 2a_o^* g_2 - 2g_{1e}b_e, \\ g_3' = -2a_o^* g_3 - 2g_{1e}c_e, \\ g_{1e}(o) = 1, \end{cases}$$

where g_2 and g_3 are odd functions.

Remark 3. By (24), we get that if

$$a_e^* = c_o = b_o = 0, \qquad \lim_{t\to 0} \frac{b_e - c_e}{a_o^*} = 0,$$
$$\left(\frac{b_e - c_e}{2a_o^*}\right)' = (c_e + b_e)\left(\left(\frac{b_e - c_e}{2a_o^*}\right)^2 - 1\right),$$

then $G = \begin{pmatrix} g_{1e} & g_2 \\ g_2 & g_{1e} \end{pmatrix}$ is the **RM** of system (11), where $g_{1e} = \mathrm{e}^{\int_0^t \frac{c_e^2 - b_e^2}{2a_o^*}\mathrm{d}t}$, $g_2 = \frac{b_e - c_e}{2a_o^*} g_{1e}$.

Lemma 2. *Suppose that matrix (10) is the **RM** of system (11), and system (11) is periodic with period 2ω. Then*

(1) *If $g_{1e}(-\omega) \neq 1$, system (11) has a unique 2ω-periodic solution which is stable when $|g_{1e}(-\omega)| < 1$ and unstable when $|g_{1e}(-\omega)| > 1$;*
(2) *If $g_{1e}(-\omega) = 1$, system (11) has infinite 2ω-periodic solutions;*
(3) *If $G(-\omega) = E$, all the solutions of system (11) are 2ω-periodic.*

Proof. By the preceding introduction, the Poincaré mapping of the periodic system (11) is $T(x, y) = G(-\omega)(x, y)^{\mathrm{T}}$, and then the solution $(x(t), y(t))$ of (11) is 2ω-periodic if and only if $(x(-\omega), y(-\omega))$ is a solution of equation

$$(G(-\omega) - E)(x, y)^{\mathrm{T}} = 0.$$

Since its determinant is $\det(G(-\omega) - E) = 2(1 - g_{1e}(-\omega))$, it is easy to deduce that the system (11) has a unique solution if and only if $\det(G(-\omega) - E) \neq 0$, i.e., $g_{1e}(-\omega) \neq 1$. Otherwise, there will exist infinitely many solutions and when $G(-\omega) = E$ all solutions of (11) are 2ω-periodic. Applying the theorem [1,p284], we know that the 2ω-periodic solution of (11) is stable when $|\operatorname{tr} G(-\omega)| = 2|g_{1e}(-\omega)| < 2$, i.e., $|g_{1e}(-\omega)| < 1$ and unstable when $|g_{1e}(-\omega)| > 1$. □

Therefore, the present lemma is true.

Theorem 1. *Assume that there exists an odd continuously differentiable function β satisfying the following conditions:*

(1) $\frac{\rho_1}{\rho_2}, \frac{\rho_3}{\rho_2}$ *are continuously differentiable and* $\lim_{t\to 0} \frac{\rho_1}{\rho_2} = 0$;
(2) $\frac{\varsigma_1}{\varsigma_3}$ *is continuously differentiable and* $\lim_{t\to 0} \frac{\varsigma_1}{\varsigma_3} = 0$;
(3)

$$\left(\frac{\varsigma_1}{\varsigma_3}\right)' = -(h_1 + 2a_o^*)\frac{\varsigma_1}{\varsigma_3} - (c\mathrm{e}^{\beta} + \bar{c}).$$

*Then $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_1\mathrm{e}^{\beta} \end{pmatrix}$ is the **RM** of system (11), where*

$$g_1 = \mathrm{e}^{\int_0^t h_1(s)\mathrm{d}s}, \qquad g_3 = \frac{\varsigma_1}{\varsigma_3} g_1, \qquad g_2 = -\frac{\rho_1}{\rho_2} g_1 - \frac{\rho_3}{\rho_2} g_3,$$
$$h_1 = c\frac{\rho_1}{\rho_2} + c\frac{\rho_3 \varsigma_1}{\rho_2 \varsigma_3} - \bar{b}\frac{\varsigma_1}{\varsigma_3} - 2a_e^*,$$
$$\varsigma_1 = \left(\frac{\rho_1}{\rho_2}\right)' - 2a^*\frac{\rho_1}{\rho_2} + c\left(\frac{\rho_1}{\rho_2}\right)^2 - \frac{\rho_3}{\rho_2}(c\mathrm{e}^{\beta} + \bar{c}) - (b + \bar{b}\mathrm{e}^{\beta}),$$
$$\varsigma_3 = -\left(\frac{\rho_3}{\rho_2}\right)' + \bar{b}\frac{\rho_1}{\rho_2} - c\frac{\rho_1\rho_3}{\rho_2^2} + 4a_o^*\frac{\rho_3}{\rho_2},$$
$$\rho_1 = \mathrm{e}^{\beta}(\beta' - 4a_e^*), \qquad \rho_2 = \bar{c} - c\mathrm{e}^{\beta}, \qquad \rho_3 = b - \bar{b}\mathrm{e}^{\beta}.$$

Besides this, if the system (11) is 2ω-periodic, then the conclusions of Lemma 2 hold.

Proof. To prove the present theorem, it is necessary to verify the matrix $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_1\mathrm{e}^{\beta} \end{pmatrix}$ satisfying the relations (23).

As

$$\rho_1 = \mathrm{e}^{\beta}(\beta' - 4a_e^*), \qquad \rho_2 = \bar{c} - c\mathrm{e}^{\beta}, \qquad \rho_3 = b - \bar{b}\mathrm{e}^{\beta}, \qquad \beta + \bar{\beta} = 0,$$

then

$$\bar{\rho}_1 = \mathrm{e}^{-2\beta}\rho_1, \qquad \bar{\rho}_2 = -\mathrm{e}^{-\beta}\rho_2, \qquad \bar{\rho}_3 = -\mathrm{e}^{\beta}\rho_3, \qquad \bar{\varsigma}_1 = \mathrm{e}^{-\beta}\varsigma_1, \qquad \bar{\varsigma}_3 = -\varsigma_3.$$

It implies that

$$h_1+\bar{h}_1=-4a_e^*-\rho_1 e^{\beta}=-\beta',$$

and $\bar{g}_1=g_1 e^{\beta}$.

Applying

$$g_1=e^{\int_0^t h_1(s)ds},\qquad g_3=\frac{\varsigma_1}{\varsigma_3}g_1,\qquad g_2=-\frac{\rho_1}{\rho_2}g_1-\frac{\rho_3}{\rho_2}g_3,$$

we get $g_1'=P_1$ and $\bar{g}_1'=(g_1e^{\beta})'=-\bar{P}_1$. Using condition (3) of the present theorem, we get $g_3'=P_3$. Using

$$\varsigma_3 g_3=\varsigma_1 g_1,\qquad g_2\rho_2+\rho_1 g_1+\rho_3 g_3=0$$

yields

$$P_2=-\left(\frac{\rho_1}{\rho_2}\right)'g_1-\left(\frac{\rho_3}{\rho_2}\right)'g_3-\frac{\rho_1}{\rho_2}P_1-\frac{\rho_3}{\rho_2}P_3,$$

and

$$g_2'-P_2=-\frac{\rho_1}{\rho_2}(g_1'-P_1)-\frac{\rho_3}{\rho_2}(g_3'-P_3)=0.$$

Thus, the matrix G is the **RM** of system (11).

When the system (11) is a 2ω-periodic system, its Poincaré mapping is $T(x,y)=G(-\omega)(x,y)^{\mathrm{T}}$. Applying Lemma 2, it implies that the present theorem is true. □

Taking $\beta=0$ in Theorem 1, it yields the following corollary.

Corollary 1.1. *Suppose that*

(1) $\frac{a_e^*}{c_o}$, $\frac{b_o}{c_o}$ *are continuously differentiable and* $\lim_{t\to 0}\frac{a_e^*}{c_o}=0$;
(2) $\frac{\lambda_1}{\lambda_3}$ *is continuously differentiable and* $\lim_{t\to 0}\frac{\lambda_1}{\lambda_3}=0$;
(3)

$$\left(\frac{\lambda_1}{\lambda_3}\right)'=-(h_2+2a_o^*)\frac{\lambda_1}{\lambda_3}-2c_e.$$

Then $G=\begin{pmatrix} g_{1e} & g_2\\ g_3 & g_{1e}\end{pmatrix}$ is the **RM** of system (11), where

$$g_{1e}=e^{\int_0^t h_2(s)ds},\qquad g_3=\frac{\lambda_1}{\lambda_3}g_{1e},\qquad g_2=-2\frac{a_e^*}{c_o}g_{1e}+\frac{b_o}{c_o}g_3,$$

$$h_2=-\frac{\lambda_1}{\lambda_3}\left(b_e+c_e\frac{b_o}{c_o}\right)+2c_e\frac{a_e^*}{c_o},$$

$$\lambda_1=4\frac{a_o^*a_e^*}{c_o}+2b_e-2c_e\frac{b_o}{c_o}-4c_e\left(\frac{a_e^*}{c_o}\right)^2-2\left(\frac{a_e^*}{c_o}\right)',$$

$$\lambda_3=4a_o^*\frac{b_o}{c_o}-2b_e\frac{a_e^*}{c_o}-2c_e\frac{a_e^*b_o}{c_o^2}-\left(\frac{b_o}{c_o}\right)'.$$

Besides this, let the system (11) be 2ω-periodic. Then, if $\int_0^{-\omega}h_2(s)ds\neq 0$, system (11) has a unique 2ω-periodic solution which is stable when $\int_0^{-\omega}h_2(s)ds<0$ and unstable when $\int_0^{-\omega}h_2(s)ds>0$. If $\int_0^{-\omega}h_2(s)ds=0$, system (11) has infinite 2ω-periodic solutions. If $G(-\omega)=E$, all the solutions of system (11) are 2ω-periodic.

Similarly, we could obtain the following theorem.

Theorem 2. *Assume that there exists an odd continuously differentiable function β satisfying the following conditions*

(1) $\frac{\rho_1}{\rho_3}$, $\frac{\rho_2}{\rho_3}$ *are continuously differentiable and* $\lim_{t\to 0}\frac{\rho_1}{\rho_3}=0$;
(2) $\frac{\sigma_1}{\sigma_2}$ *is continuously differentiable and* $\lim_{t\to 0}\frac{\sigma_1}{\sigma_2}=0$;
(3)

$$\left(\frac{\sigma_1}{\sigma_2}\right)'=(-h_3+2a_o^*)\frac{\sigma_1}{\sigma_2}-(b+\bar{b}e^{\beta}).$$

Then $G=\begin{pmatrix} g_1 & g_2 \\ g_3 & g_1 e^{\beta}\end{pmatrix}$ is the **RM** of system (11), where

$$g_1 = e^{\int_0^t h_3(s)ds}, \qquad g_2=\frac{\sigma_1}{\sigma_2}g_1, \qquad g_3=-\frac{\rho_1}{\rho_3}g_1-\frac{\rho_2}{\rho_3}g_2,$$
$$h_3=\bar{b}\frac{\rho_1}{\rho_3}+\bar{b}\frac{\rho_2\sigma_1}{\rho_3\sigma_2}-c\frac{\sigma_1}{\sigma_2}-2a_e^*,$$
$$\sigma_1=\left(\frac{\rho_1}{\rho_3}\right)'-2\bar{a}^*\frac{\rho_1}{\rho_3}+\bar{b}\left(\frac{\rho_1}{\rho_3}\right)^2-\frac{\rho_2}{\rho_3}(b+\bar{b}e^{\beta})-(ce^{\beta}+\bar{c}),$$
$$\sigma_2=-\left(\frac{\rho_2}{\rho_3}\right)'+c\frac{\rho_1}{\rho_3}-\bar{b}\frac{\rho_1\rho_2}{\rho_3^2}-4a_o^*\frac{\rho_2}{\rho_3},$$
$$\rho_1=e^{\beta}(\beta'-4a_e^*), \qquad \rho_2=\bar{c}-ce^{\beta}, \qquad \rho_3=b-\bar{b}e^{\beta}.$$

Besides this, let system (8) be 2ω-periodic. Then the results of Lemma 2 are true.

Taking $\beta=0$ in Theorem 2, we get the following corollary.

Corollary 2.1. *Suppose that*

(1) $\frac{a_e^*}{b_o}$, $\frac{c_o}{b_o}$ *are continuously differentiable and* $\lim_{t\to 0}\frac{a_e^*}{b_o}=0$;
(2) $\frac{\mu_1}{\mu_2}$ *is continuously differentiable and* $\lim_{t\to 0}\frac{\mu_1}{\mu_2}=0$;
(3)

$$\left(\frac{\mu_1}{\mu_2}\right)'=(-h_4+2a_o^*)\frac{\mu_1}{\mu_2}-2b_e;$$

Then $G=\begin{pmatrix} g_{1e} & g_2 \\ g_3 & g_{1e}\end{pmatrix}$ is the **RM** of system (11), where

$$g_{1e}=e^{\int_0^t h_4(s)ds}, \qquad g_2=\frac{\mu_1}{\mu_2}g_{1e}, \qquad g_3=2\frac{a_e^*}{b_o}g_{1e}+\frac{c_o}{b_o}g_2,$$
$$h_4=-\frac{\mu_1}{\mu_2}\left(c_e+b_e\frac{c_o}{b_o}\right)-2b_e\frac{a_e^*}{b_o},$$
$$\mu_1=4\frac{a_o^*a_e^*}{b_o}+2c_e-2b_e\frac{c_o}{b_o}-4b_e\left(\frac{a_e^*}{b_o}\right)^2+2\left(\frac{a_e^*}{b_o}\right)',$$
$$\mu_2=-4a_o^*\frac{c_o}{b_o}+2c_e\frac{a_e^*}{b_o}+2b_e\frac{a_e^*c_o}{b_o^2}-\left(\frac{c_o}{b_o}\right)'.$$

Besides this, let system (11) be 2ω-periodic. Then if $\int_0^{-\omega}h_4(s)ds\neq 0$, system (11) has a unique 2ω-periodic solution which is stable when $\int_0^{-\omega}h_4(s)ds<0$ and unstable when $\int_0^{-\omega}h_4(s)ds>0$. If $\int_0^{-\omega}h_4(s)ds=0$, system (11) has infinite 2ω-periodic solutions. If $G(-\omega)=E$, all the solutions of system (11) are 2ω-periodic.

Example 1. Differential system

$$\begin{pmatrix} x \\ y\end{pmatrix}'=\begin{pmatrix} \frac{1}{2}\sin^3 t\cos t-\sin^2 t & -\frac{1}{2}\cos^3 t-2\sin t \\ \frac{1}{2}\cos t(\sin^4 t+\sin^2 t+2)-2\sin t & -\frac{1}{2}\sin^3 t\cos t+\sin^2 t\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} \tag{25}$$

has the **RM**:

$$G(t)=\begin{pmatrix} 1+\sin^2 t & \sin t \\ \sin t(2+\sin^2 t) & 1+\sin^2 t\end{pmatrix}.$$

It is not difficult to verify that all the conditions of Corollary 1.1 are satisfied for system (25), in which $a_e^*=-\sin^2 t$, $a_o^*=\frac{1}{2}\sin^3 t\cos t$, $b_e=-\frac{\cos^3 t}{2}$, $b_o=-2\sin t$, $c_e=-\frac{1}{2}\cos t(2+\sin^2 t+\sin^4 t)$, $c_o=-2\sin t$. By simple computations, we obtain $\frac{a_e^*}{c_o}=\frac{\sin t}{2}$, $\frac{b_o}{c_o}=1$, $\lim_{t\to 0}\frac{a_e^*}{c_o}=0$, $(\frac{a_e^*}{c_o})'=\frac{\cos t}{2}$, $(\frac{b_o}{c_o})'=0$, $\lambda_1=\frac{1}{2}\cos t\sin^2 t(\sin^2 t+2)(\sin^2+3)$, $\lambda_3=\frac{1}{2}\cos t\sin t(1+\sin^2 t)(3+\sin^2 t)$, $\frac{\lambda_1}{\lambda_3}=\frac{\sin t(2+\sin^2 t)}{1+\sin^2 t}$, $\lim_{t\to 0}\frac{\lambda_1}{\lambda_3}=0$, $h_2=\frac{2\sin t\cos t}{1+\sin^2 t}$, $g_{1e}=e^{\int_0^t h_2(s)ds}=1+\sin^2 t$, $g_3=\frac{\lambda_1}{\lambda_3}g_{1e}=\sin t(2+\sin^2 t)$, $g_2=-\frac{2a_e^*}{c_o}g_{1e}+\frac{b_o}{c_o}g_3=\sin t$. In view of $G(-\pi)=E$ and Corollary 1.1, it implies that all the solutions defined on $[-\pi,\pi]$ of system (25) are 2π-periodic.

Example 2. Differential system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = -\frac{1}{2}\begin{pmatrix} (1+2S^2-S^3+S^4)T & Te^{-S}(1+S-S^2+S^3) \\ Te^{S}(2-2S+S^2-3S^3+S^4-S^5) & -T(1+2S^2-S^3+S^4) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{26}$$

has the **RM**

$$G(t) = \begin{pmatrix} (1+S^2)e^{S} & S \\ S(2+S^2) & (1+S^2)e^{-S} \end{pmatrix},$$

where $S := \sin t$, $T := \cos t$.

It is not difficult to check that all the conditions of Theorem 2 are satisfied for system (26), in which $a_e^* = -\frac{1}{2}T(1+S^2)^2$, $a_o^* = \frac{1}{2}TS^3$, $b = -\frac{1}{2}Te^{-S}(1+S-S^2+S^3)$, $c = -\frac{1}{2}Te^{S}(2-2S+S^2-3S^3+S^4-S^5)$, $\beta = -2S$, $\rho_1 = 2TS^2(2+S^2)e^{-2S}$, $\rho_2 = -ST(1+S^2)(2+S^2)e^{-S}$, $\rho_3 = -TS(1+S^2)e^{-S}$. By simple computations, we get $\frac{\rho_1}{\rho_3} = -2Se^{-S}\frac{2+S^2}{1+S^2}$, $\frac{\rho_2}{\rho_3} = 2+S^2$, $\sigma_1 = -\frac{2TS^2e^{-S}}{1+S^2}$, $\sigma_2 = -2ST$.$h_4 = \frac{T(1+S)^2}{1+S^2}$, $g_1 = e^{\int_0^t h_4(s)ds} = (1+S^2)e^{S}$, $g_2 = \frac{\sigma_1}{\sigma_2}g_1 = S$, $g_3 = -\frac{\rho_1}{\rho_3}g_1 - \frac{\rho_2}{\rho_3}g_2 = S(2+S^2)$, $\bar{g}_1 = g_1 e^{\beta}$.

Since this system is 2π-periodic and $G(-\pi) = E$. By Theorem 2, all the solutions defined on $[-\pi, \pi]$ of system (26) are 2π-periodic.

Theorem 3. *Assume that there exists an even function $k(t)$ satisfying the following conditions*

(1) $\frac{\delta_1}{\delta_2}$ *is continuously differentiable and* $\lim_{t\to 0}\frac{\delta_1+\bar{\delta}_1}{\delta_2} = 0$;
(2) $\frac{m_1}{\bar{m}_1}$ *is continuously differentiable and* $\lim_{t\to 0}\frac{m_1}{\bar{m}_1} = -1$;
(3)

$$\left(\frac{m_1}{\bar{m}_1}\right)' = -(h_5+\bar{h}_5)\frac{m_1}{\bar{m}_1}.$$

Then $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & \bar{g}_1 \end{pmatrix}$ is the **RM** of system (11), where

$$g_1 = e^{\int_0^t h_5(s)ds}, \qquad g_2 = -\left(\frac{\delta_1}{\delta_2} - \frac{\bar{\delta}_1}{\delta_2}\frac{m_1}{\bar{m}_1}\right)g_1,$$

$$h_5 = -2a_e^* + (\bar{b}k + c)\left(\frac{\delta_1}{\delta_2} - \frac{\bar{\delta}_1 m_1}{\delta_2 \bar{m}_1}\right),$$

$$m_1 = -\left(\frac{\delta_1}{\delta_2}\right)' + 2\bar{a}^*\frac{\delta_1}{\delta_2} - \frac{\delta_1}{\delta_2}\left(c\frac{\delta_1}{\delta_2} + \bar{c}\frac{\bar{\delta}_1}{\delta_2}\right) + k\left(\bar{b}\frac{\delta_1}{\delta_2} + b\frac{\delta_1}{\delta_2}\right) + b,$$

$$\delta_1 = \bar{c} - kb; \qquad \delta_2 = k' + 4a_o^* k.$$

Besides this, let system (11) be 2ω-periodic. Then the conclusions of Lemma 2 are true.

Proof. To prove this theorem, it is necessary to check the matrix $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & \bar{g}_1 \end{pmatrix}$ satisfying the identity (23). As

$$g_1 = e^{\int_0^t h_5(s)ds}, \qquad g_2 = -\left(\frac{\delta_1}{\delta_2} - \frac{\bar{\delta}_1}{\delta_2}\frac{m_1}{\bar{m}_1}\right)g_1, \qquad g_3 = kg_2,$$

$$h_5 = -2a_e^* + (\bar{b}k + c)\left(\frac{\delta_1}{\delta_2} - \frac{\bar{\delta}_1 m_1}{\delta_2 \bar{m}_1}\right).$$

Then $P_1 = -2a_e^* g_1 - g_2 c - g_3 \bar{b} = h_5 g_1$, and $g_1' = h_5 g_1 = P_1$.

By the condition (3) of the present theorem, it implies that

$$\bar{g}_1 = -\frac{m_1}{\bar{m}_1}g_1 = -\frac{m_1}{\bar{m}_1}e^{\int_0^t h_5 ds},$$

and $\bar{g}_1' = -\bar{P}_1$.

Applying

$$g_2 = -\left(\frac{\delta_1}{\delta_2} - \frac{\bar{\delta}_1}{\delta_2}\frac{m_1}{\bar{m}_1}\right)g_1, \qquad m_1 g_1 + \bar{m}_1 \bar{g}_1 = 0,$$

it implies

$$P_2 = -\left(\frac{\delta_1}{\delta_2}\right)' g_1 - \left(\frac{\bar{\delta}_1}{\delta_2}\right)' \bar{g}_1 - \frac{\delta_1}{\delta_2} P_1 + \frac{\bar{\delta}_1}{\delta_2} \bar{P}_1,$$

so

$$g_2' - P_2 = -\frac{\delta_1}{\delta_2}(g_1' - P_1) - \frac{\bar{\delta}_1}{\delta_2}(\bar{g}_1' + \bar{P}_1) = 0,$$

that is $g_2' = P_2$.

Using

$$\delta_2 g_2 + \delta_1 g_1 + \bar{\delta}_1 \bar{g}_1 = 0,$$

we get

$$P_3 = k' g_2 + k P_2.$$

Thus

$$g_3' - P_3 = k(g_2' - P_2) = 0.$$

Therefore, matrix G is the **RM** of system (11). By Lemma 2, the proof of the present theorem is complete. □

Taking $k = -1$ in Theorem 3, it yields the following corollary.

Corollary 3.1. *For system (11), suppose that the following conditions are satisfied*

(1) $\frac{b_e+c_e}{a_o^*}$, $\frac{b_o-c_o}{a_o^*}$ *are continuously differentiable and* $\lim_{t\to 0} \frac{b_e+c_e}{a_o^*} = 0$;

(2) $\frac{w_1}{w_0}$ *is continuously differentiable and* $\lim_{t\to 0} \frac{w_1}{w_0} = 0$;

(3)

$$\left(\frac{w_1}{w_0}\right)' = -\left(\frac{b_o^2 - c_o^2}{2a_o^*} + h_6\right)\frac{w_1}{w_0} + \frac{b_e + c_e}{2a_o^*}(b_o + c_o) + 2a_e^*.$$

Then $G = \begin{pmatrix} g_{1e} + g_{1o} & g_2 \\ -g_2 & g_{1e} - g_{1o} \end{pmatrix}$ is the **RM** of system (11), where

$$g_{1e} = \mathrm{e}^{\int_0^t h_6(s)\mathrm{d}s}, \qquad g_{1o} = -\frac{w_1}{w_0} g_{1e}, \qquad g_2 = \left(\frac{b_e + c_e}{2a_o^*} - \frac{b_o - c_o}{2a_o^*}\frac{w_1}{w_0}\right) g_{1e},$$

$$h_6 = -2a_e^* \frac{w_1}{w_o} - (c_e - b_e)\left(\frac{b_e + c_e}{2a_o^*} - \frac{b_o - c_o}{2a_o^*}\frac{w_1}{w_0}\right),$$

$$w_0 = \left(\frac{b_o - c_o}{2a_o^*}\right)' - (b_e + c_e)\frac{a_e^*}{a_o^*} - \frac{c_e^2 - b_e^2}{4a_o^{*2}}(b_o - c_o) - \frac{(b_o - c_o)^2}{4a_o^{*2}}(b_o + c_o) + b_o + c_o,$$

$$w_1 = \left(\frac{b_e + c_e}{2a_o^*}\right)' - (b_o - c_o)\frac{a_e^*}{a_o^*} + \frac{c_o^2 - b_o^2}{4a_o^{*2}}(b_e + c_e) - \frac{(b_e + c_e)^2}{4a_o^{*2}}(c_e - b_e) + b_e - c_e.$$

Besides this, let system (11) be 2ω-periodic. Then if $\int_0^{-\omega} h_6(s)\mathrm{d}s \neq 0$, system (11) has a unique 2ω-periodic solution which is stable when $\int_0^{-\omega} h_6(s)\mathrm{d}s < 0$ and unstable when $\int_0^{-\omega} h_6(s)\mathrm{d}s > 0$. If $\int_0^{-\omega} h_6(s)\mathrm{d}s = 0$, system (11) has infinite 2ω-periodic solutions. If $G(-\omega) = E$, all the solutions of system (11) are 2ω-periodic.

Proof. To prove the present conclusion, we only need to verify matrix G satisfying identity (24). Indeed, as

$$g_{1e} = \mathrm{e}^{\int_0^t h_6(s)\mathrm{d}s}.$$

So

$$g_{1e}' = h_6 g_{1e}.$$

Using

$$g_{1o} = -\frac{w_1}{w_0} g_{1e}, \qquad g_2 = \left(\frac{b_e + c_e}{2a_o^*} - \frac{b_o - c_o}{2a_o^*}\frac{w_1}{w_0}\right) g_{1e},$$

$$h_6 = -2a_e^* \frac{w_1}{w_o} - (c_e - b_e)\left(\frac{b_e + c_e}{2a_o^*} - \frac{b_o - c_o}{2a_o^*}\frac{w_1}{w_0}\right),$$

yields

$$Q_1 = -2a_e^* g_{1o} - g_2 c_e - b_e g_3 = h_6 g_{1e},$$

hence $g_{1e}' = Q_1$.

As

$$g_{1o} = -\frac{w_1}{w_0} g_{1e},$$

so

$$g_{1o}' = -\left(\left(\frac{w_1}{w_0}\right)' + \frac{w_1}{w_0} h_6\right) g_{1e}.$$

Using the condition (3) of the present corollary, we obtain

$$Q_0 = -2a_e^* g_{1e} - g_2 c_o + g_3 b_o = -\frac{w_1}{w_0} g_{1e} = g_{1o}'.$$

Applying

$$w_1 g_{1e} + w_0 g_{1o} = 0$$

yields

$$P_2 = \left(\frac{b_e + c_e}{2a_o^*}\right)' g_{1e} + \left(\frac{b_o - c_o}{2a_o^*}\right)' g_{1o} + \frac{b_e + c_e}{2a_o^*} Q_1 + \frac{b_o - c_o}{2a_o^*} Q_0.$$

Thus

$$g_2' - P_2 = \frac{b_e + c_e}{2a_o^*}(g_{1e}' - Q_1) + \frac{b_o - c_o}{2a_o^*}(g_{1o}' - Q_0) = 0.$$

Using

$$g_2 = \frac{b_e + c_e}{2a_o^*} g_{1e} + \frac{b_o - c_o}{2a_o^*} g_{1o},$$

implies

$$Q_2 = -Q_3,$$

and

$$g_3' - Q_3 = -(g_2' - Q_2) = 0.$$

Therefore, matrix G satisfies the relations (24), i.e., matrix G is the **RM** of system (11).

When the system (11) is 2ω-periodic system, its Poincaré mapping is $T(x, y) = G(-\omega)(x, y)^{\mathrm{T}}$. As $g_{1e}(-\omega) = \mathrm{e}^{\int_0^{-\omega} h_6(s)\mathrm{d}s}$, so, $g_{1e}(-\omega) = 1$ is equivalent to $\int_0^{-\omega} h_6(s)\mathrm{d}s = 0$. Applying Lemma 2 yields that the present corollary is true. $\square$

Similarly, we obtain the following theorem.

Theorem 4. *Assume that there exists an even function $k(t)$ satisfying the following conditions*

(1) $\frac{\epsilon_1}{\epsilon_3}$ *is continuously differentiable and* $\lim_{t\to 0} \frac{\epsilon_1}{\epsilon_3} = 0$;

(2) $\frac{n_1}{\bar{n}_1}$ *is continuously differentiable and* $\lim_{t\to 0} \frac{n_1}{\bar{n}_1} = -1$;

(3)

$$\left(\frac{n_1}{\bar{n}_1}\right)' = -(h_7 + \bar{h}_7)\frac{n_1}{\bar{n}_1}.$$

Then $G = \begin{pmatrix} g_1 & g_2 \\ g_3 & \bar{g}_1 \end{pmatrix}$ is the **RM** of system (11), where

$$g_1 = \mathrm{e}^{\int_0^t h_7(s)\mathrm{d}s}, \qquad g_3 = -\left(\frac{\epsilon_1}{\epsilon_3} - \frac{\bar{\epsilon}_1}{\epsilon_3}\frac{n_1}{\bar{n}_1}\right) g_1, \qquad g_2 = k g_3,$$

$$h_7 = -2a_e^* + (\bar{b} + kc)\left(\frac{\epsilon_1}{\epsilon_3} - \frac{\bar{\epsilon}_1 n_1}{\epsilon_3 \bar{n}_1}\right),$$

$$n_1 = \left(\frac{\epsilon_1}{\epsilon_3}\right)' - 2\bar{a}^*\frac{\epsilon_1}{\epsilon_3} + \left(\frac{\epsilon_1}{\epsilon_3}\right)^2 (ck + \bar{b}) + \frac{\epsilon_1 \bar{\epsilon}_1}{\epsilon_2^2}(\bar{c}k + b) - \bar{c},$$

$$\epsilon_1 = \bar{c}k - b, \qquad \epsilon_3 = -k' + 4a_o^* k.$$

Besides this, let system (8) be 2ω-periodic. Then the results of Lemma 2 hold.

Taking $k(t) = 1$ in Theorem 4, it implies the following corollary.

Corollary 4.1. *Suppose that the following conditions are satisfied*

(1) $\frac{b_e-c_e}{a_o^*}$, $\frac{b_o+c_o}{a_o^*}$ *are continuously differentiable and* $\lim_{t\to 0}\frac{b_e-c_e}{a_o^*} = 0$;

(2) $\frac{u_1}{u_0}$ *is continuously differentiable and* $\lim_{t\to 0}\frac{u_1}{u_0} = 0$;

(3)

$$\left(\frac{u_1}{u_0}\right)' = \left(\frac{b_o^2-c_o^2}{2a_o^*} - h_8\right)\frac{u_1}{u_0} - \frac{b_e-c_e}{2a_o^*}(b_o-c_o) + 2a_e^*.$$

Then $G = \begin{pmatrix} g_{1e}+g_{1o} & g_2 \\ g_2 & g_{1e}-g_{1o}\end{pmatrix}$ is the **RM** of system (11), where

$$g_{1e} = \mathrm{e}^{\int_0^t h_8(s)\mathrm{d}s}, \qquad g_{1o} = -\frac{u_1}{u_0}g_{1e}, \qquad g_2 = \left(\frac{b_e-c_e}{2a_o^*} - \frac{b_o+c_o}{2a_o^*}\frac{u_1}{u_0}\right)g_{1e},$$

$$h_8 = 2a_e^*\frac{u_1}{u_o} - (c_e+b_e)\left(\frac{b_e-c_e}{2a_o^*} - \frac{b_o+c_o}{2a_o^*}\frac{u_1}{u_0}\right),$$

$$u_0 = \left(\frac{b_o+c_o}{2a_o^*}\right)' - (b_e-c_e)\frac{a_e^*}{a_o^*} - \frac{b_e^2-c_e^2}{4a_o^{*2}}(b_o+c_o) - \frac{(b_o+c_o)^2}{4a_o^{*2}}(c_o-b_o) + b_o - c_o,$$

$$u_1 = \left(\frac{b_e-c_e}{2a_o^*}\right)' - (b_o+c_o)\frac{a_e^*}{a_o^*} - \frac{c_o^2-b_o^2}{4a_o^{*2}}(b_e-c_e) - \frac{(b_e-c_e)^2}{4a_o^{*2}}(c_e+b_e) + b_e + c_e.$$

Besides this, let system (11) be 2ω-periodic. Then if $\int_0^{-\omega} h_8(s)\mathrm{d}s \neq 0$, system (11) has a unique 2ω-periodic solution which is stable when $\int_0^{-\omega} h_8(s)\mathrm{d}s < 0$ and unstable when $\int_0^{-\omega} h_8(s)\mathrm{d}s > 0$. If $\int_0^{-\omega} h_8(s)\mathrm{d}s = 0$, system (11) has infinite 2ω-periodic solutions. If $G(-\omega) = E$, all the solutions of system (11) are 2ω-periodic.

Remark 4. Summarizing the above. If one of above the theorems or corollaries holds, then $F(t, x) = \mathrm{e}^{\alpha}G(t)x$ is the **RF** of systems (6)–(8). Besides this, these systems are 2ω-periodic. Then if $g_{1e}(\omega) \neq \frac{\mathrm{e}^{\alpha(\omega)}+\mathrm{e}^{-\alpha(\omega)}}{2}$, each of them has a unique 2ω-periodic solution which is stable when $|g_{1e}(\omega)\mathrm{e}^{-\alpha(\omega)}| < 1$ and unstable when $|g_{1e}(\omega)\mathrm{e}^{-\alpha(\omega)}| > 1$. If $g_{1e}(\omega) = \frac{\mathrm{e}^{\alpha(\omega)}+\mathrm{e}^{-\alpha(\omega)}}{2}$, each of them has infinite 2ω-periodic solutions. If $F(-\omega) = \mathrm{e}^{-\alpha(\omega)}G(-\omega) = E$, all the solutions of these systems are 2ω-periodic.

Remark 5. In Corollary 3.1, taking $w_1 \equiv 0$, implies $g_{1o} = 0$, $(b_e+c_e)(b_o+c_o) = -4a_e^*a_o^*$, $g_2 = \frac{b_e+c_e}{2a_o^*}g_{1e}$, $(\frac{b_e+c_e}{2a_o^*})' = (c_e-b_e)(1+(\frac{b_e+c_e}{2a_o^*})^2)$. In this case, $G = \begin{pmatrix} g_{1e} & g_2 \\ -g_2 & g_{1e}\end{pmatrix}$ is the **RM** of system (11). That is Theorem 1 of [8].

Obtained results for linear differential system can be extended for nonlinear systems with small parameter.

Consider the nonlinear differential system depending on parameter ϵ.

$$x' = f(t, x, \epsilon), \quad t \in R, x \in D \subset R^n, \tag{27}$$

where f is a continuous 2ω-periodic vector function for all t, small $|\epsilon|$, and also continuously differentiable with respect to components of vector x. Let $x = \eta_0(t)$ be a 2ω-periodic solution of the system (27) in which $\epsilon = 0$.

Using the concept of a reflecting matrix we can reformulate the following theorem.

Theorem 5. *Let matrix $F(t)$ be the **RM** of the linear system (9) with matrix $A(t) = \frac{\partial f}{\partial x}(t, \eta_0(t), 0)$. If there is no unit among solutions μ_i of equation* $\det(F(-\omega) - \mu E) = 0$, *then system (27) with sufficiently small $|\epsilon|$ has a unique 2ω-periodic solution $x = x(t, \epsilon)$ with initial point $x(0, \epsilon)$ close to $\eta_0(0)$. Besides, $x(t, \epsilon)$ is a continuous function with respect to (t, ϵ), and $x(t, 0) = \eta_0(t)$. If, moreover, f is continuously differentiable with respect to ϵ, then $x(t, \epsilon)$ is also continuously differentiable.*

Proof. Since multiplicators μ_i for 2ω-periodic linear system with the **RM** $F(t)$ are solutions of the equation $\det(F(-\omega) - \mu E) = 0$, therefore, validity of this theorem follows from Theorem 2.3 in the book [12, p. 488]. □

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